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MULTIINDEX MITTAG-LEFFLER FUNCTIONS, RELATED GELFOND-LEONTIEV OPERATORS AND LAPLACE TYPE INTEGRAL TRANSFORMS *

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Abstract

Recently, the interest in the Mittag-Leffler functions and their popularity have increased in view of their important role and applications in fractional calculus and related integral and differential equations of fractional order. In this paper we introduce analogues of these functions, depending on two sets of multiple indices. We study their basic properties and relations with the operators of generalized fractional calculus. Corresponding generalized operators of integration and differention of the so-called Gelfond-Leontiev type, as well as Borel-Laplace type integral transforms, are also introduced and studied. This can be considered as a short survey exposition of results whose detailed proofs could be found in other author's papers.

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1. Introduction

For a long time, the Mittag-Leffler functions have been almost totally ignored in the common handbooks on special functions and existing tables of Laplace transforms, although a description of their properties has appeared yet in the third volume of the Bateman Project [8], in a chapter devoted to “miscellaneous functions”. Recently, the interest in them and their popularity have sharply increased in view of their important role in fractional calculus and related integral

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and differential equations of fractional order (as their solutions) and the applications these tools have found in treating fractional-order control systems, fractional viscoelastic models, and many other problems of physics, mechanics, practice (see e.g. [21],[12]).

The *Mittag-Leffler (M-L) functions* E_α (Mittag-Leffler, 1902-1905) and $E_{\alpha,\beta}$ (Agarwal, 1953), are defined by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha>0,\ \beta>0. \quad (1)$$

They are natural extensions of the exponential function and trigonometric functions:

$$y_1(z) = E_1(z) = \exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)},\ y_2(z) = E_2(-z^2) = \cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(2k+1)},$$

satisfying the (integer order) differential equations

$$D^1 y_1(\lambda z) = \lambda y_1(\lambda z), \quad D^2 y_2(\lambda z) = -\lambda^2 y_2(\lambda z).$$

However, in the case of M-L functions (1) with a fractional index α, one has fractional order differential equations, like:

$$D^\alpha y(z) = \lambda y(z) \quad \text{with} \quad y(z) = z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^\alpha),\ \alpha>0. \quad (2)$$

The Mittag-Leffler functions (1) are entire functions of given order $\rho = 1/\alpha$ and type $\sigma = 1$, in a sense, the simplest such functions. They provide also an important class of special functions that are H-functions, not reducible (for irrational α) to the more simple and popular Meijer G-functions, namely:

$$E_{\alpha,\beta}(z) = H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(1-\beta,\alpha) \end{matrix} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\mu-\alpha s)} (-z)^{-s} ds. \quad (3)$$

Functions (1) were studied in details by Dzrbashjan [5],[7], using alternative denotation $E_\rho(z;\mu)$ for $E_{\frac{1}{\rho},\mu}(z)$: asymptotic formulas in different parts of complex plane, kernel-functions of inverse Borel-type integral transforms, various relations and representations. Unfortunately, most of his achievements were published only in Russian. The lack of literature on M-L functions till recently, has been nowadays compensated by a long list of newly written papers, surveys and books, see for example the encyclopedy of fractional calculus [23], the articles of Kilbas and Samko [13], Gorenflo, Luchko and Rogozin [11], Gorenflo, Kilbas and Rogozin [10], the book of Podlubny [21], the surveys of Gorenflo and Mainardi [12], Luchko [18], and all the references therein, as well as the 2-years contents of this journal, *"Fractional Calculus & Applied Analysis"*, Vols. 1, 2 (1998,1999), etc.

In this paper we introduce *special functions of Mittag-Leffler type* that are *multiindex analogues of* (1), depending instead on 2 indices $\alpha := 1/\rho > 0, \beta := \mu \in R$, on 2 sets of indices $(1/\rho_1, 1/\rho_2, \dots, 1/\rho_m)$, $(\mu_1, \mu_2, \dots, \mu_m)$. In Section 2 we shortly survey their basic properties and in Section 3 we study their relations involving operators of generalized fractional calculus. Section 4 deals with a class of Gelfond-Leontiev (G-L) operators of integration and differentiation generated by the multiindex M-L functions, that happen to be also a class of operators of the generalized fractional calculus [15]. In Section 5 we consider a Laplace-type integral transform, corresponding to these G-L operators. The operators and the H-transform from Sections 4,5 generalize, among other classical operators, the so-called hyper-Bessel differential and integral operators and the Obrechkoff integral transform, see for example [15, Ch.3].

2. Multiindex Mittag-Leffler functions

Definition 1. Let $m > 1$ be an integer, $\rho_1, \ldots, \rho_m > 0$ and $\mu_1, \ldots, \mu_m$ be arbitrary real numbers. By means of the "multiindices" $(\rho_i), (\mu_i)$ we introduce the so-called *multiindex (multiple, m-tuple) Mittag-Leffler functions*

$$E_{(\frac{1}{\rho}_i),(\mu_i)}(z) = \sum_{k=0}^{\infty} \varphi_k z^k = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1) \ldots \Gamma(\mu_m + k/\rho_m)} \,. \tag{4}$$

Theorem 1. *For arbitrary sets of indices $\rho_i > 0, \mu_i \in R, i = 1, \ldots, m,$ the multiple Mittag-Leffler function* (4) *is an entire function of order*

$$\rho = \left(\sum_{i=1}^{m} \frac{1}{\rho_i} \right)^{-1}, \quad \text{i.e.} \quad \frac{1}{\rho} = \frac{1}{\rho_1} + \ldots + \frac{1}{\rho_m} \,, \tag{5}$$

and type

$$\sigma = \left(\frac{\rho_1}{\rho} \right)^{\rho/\rho_1} \ldots \left(\frac{\rho_m}{\rho} \right)^{\rho/\rho_m} . \tag{6}$$

Moreover, for every positive ε, the asymptotic estimate

$$|E_{(\frac{1}{\rho}_i),(\mu_i)}(z)| < \exp\left((\sigma + \varepsilon)|z|^{\rho}\right), \quad |z| \geq r_0 > 0, \tag{7}$$

holds with ρ, σ as in (5), (6) *for $|z| \geq r_0(\varepsilon)$, $r_0(\varepsilon)$ sufficiently large.*

The details of *Proof* (using some tecniques from [6], [10]) will appear in [17].

Next, let us emphasize the place that the multiple M-L functions occupy among the other special functions, especially in the scheme of the generalized hypergeometric functions known as Meijer's G-functions (see [8, Vol.1],[15, App.]) and Fox's H-functions (see e.g. [24], [22],[15, App.]).

Definition 2. By a *Fox's H-function* it is meant a generalized hypergeometric function, defined by means of the Mellin-Barnes type contour integral

$$H^{m,n}_{p,q} \left[\sigma \left| \begin{array}{c} (a_k, A_k)_1^p \\ (b_k, B_k)_1^q \end{array} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}'} \frac{\prod\limits_{k=1}^{m} \Gamma(b_k - B_k s) \prod\limits_{j=1}^{n} \Gamma\left(1 - a_j + s A_j\right)}{\prod\limits_{k=m+1}^{q} \Gamma(1 - b_k + s B_k) \prod\limits_{j=n+1}^{p} \Gamma\left(a_j - s A_j\right)} \sigma^s \, ds, \tag{8}$$

where $\mathcal{L}'$ is a suitable contour in C, the orders (m,n,p,q) are integers $0 \le m \le q$, $0 \le n \le p$ and the parameters $a_j \in R, A_j > 0, j = 1,\dots,p$, $b_k \in R, B_k > 0$, $k = 1,\dots,q$ are such that $A_j(b_k + l) \ne B_k(a_j - l' - 1)$, $l, l' = 0,1,2,\dots$. For various type of contours and conditions for existence and analyticity of function (8) in disks $\subset C$ whose radii are $\rho = \prod_{j=1}^{p} A_j^{-A_j} \prod_{k=1}^{q} B_k^{B_k} > 0$, one can see [22],[24],[15, App.], etc.

For $A_1 = \dots = A_p = 1, B_1 = \dots = B_q = 1$, (8) turns into more popular *Meijer's G-function* (see [8, Vol.1, Ch.5],[22],[15]). The G- and H-functions encompass almost all the elementary and special functions and this makes the knowledge on them very useful. While the generalized hypergeometric functions ${}_pF_q$ (and all their special cases) are G-functions (see [8, §5.6, Vol.1]), the M-L functions (1) with irrational indices $\alpha > 0$ and the Wright's generalized hypergeometric functions ${}_p\Psi_q$ with irrational $A_j, B_k > 0$, give examples of *H-functions, not reducible to G-functions*: see representation (3) and

$$ {}_p\Psi_q \left[\begin{array}{c} (a_1,A_1),\dots,(a_p,A_p) \\ (b_1,B_1),\dots,(b_q,B_q) \end{array} \middle| \sigma \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + kA_1)\dots\Gamma(a_p + kA_p)}{\Gamma(b_1 + kB_1)\dots\Gamma(b_q + kB_q)} \frac{\sigma^k}{k!} $$
$$ = H^{1,p}_{p,q+1} \left[-\sigma \middle| \begin{array}{c} (1-a_1,A_1),\dots,(1-a_p,A_p) \\ (0,1),(1-b_1,B_1),\dots,(1-b_q,B_q) \end{array} \right]. \qquad (9) $$

Only for $A_1 = \dots = A_p = B_1 = \dots = B_q = 1$, (9) reduces to const.${}_pF_q(\sigma)$.

LEMMA 1. *The multiple M-L functions* (4) *are Wright's generalized hypergeometric functions* (9) *as well as Fox's H-functions of the form*:

$$ E_{(\frac{1}{\rho_i}),(\mu_i)}(z) = {}_1\Psi_m \left[\begin{array}{c} (1,1) \\ (\mu_i, \frac{1}{\rho_i})_1^m \end{array} \middle| z \right] = H^{1,1}_{1,m+1} \left[-z \middle| \begin{array}{c} (0,1) \\ (0,1),(1-\mu_i, \frac{1}{\rho_i})_1^m \end{array} \right]. \qquad (10) $$

Thus, they have the following Mellin-Barnes type contour integral representation, extending integral formula (3) :

$$ E_{(\frac{1}{\rho_i}),(\mu_i)}(z) = \frac{1}{2\pi i} \int_{\mathcal{L}'} \frac{\Gamma(-s')\Gamma(1+s')}{\prod_{i=1}^{m} \Gamma(\mu_i + \frac{s'}{\rho_i})} (-z)^{s'} ds' = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\Gamma(1-s)}{\prod_{i=1}^{m} \Gamma(\mu_i - \frac{s}{\rho_i})} (-z)^{-s} ds, \qquad (11) $$

$z \ne 0$, *where $\mathcal{L}$ is any contour in C running from $-i\infty$ to $+i\infty$ in a way that the poles $s = 0,-1,-2,\dots$ of $\Gamma(s)$ lie to the left of $\mathcal{L}$ and the poles $s = 1,2,\dots$ of $\Gamma(1-s)$ - to the right of it.*

P r o o f. Comparing definitions (4) and (9), we obtain (10). On the other hand, the second equality in (10) and definition (8) of the H-functions yield the integral representation (11), for which the details on the contour $\mathcal{L}$ can be seen, e.g. in [24, p.11].

The representation (10) of multiinex M-L functions (4) as Fox's H- functions, allow to describe their asymptotic behaviour as $z \to 0, z \to \infty$. In the case of

M-L function (1) ($m = 1$), Dzrbashjan [5],[7] established different asymptotic formulas for $|z| \to \infty$, valid in different parts of the complex plain and under different conditions on ρ, μ. For example, if $\rho > 1/2$ and inside angle domains, (1) is $\approx \rho z^{\rho(1-\mu)} \exp(z^\rho)$. An asymptotic estimate in the case $m > 1$ is given by (7), and more details follow by interpreting the multiindex M-L functions as ${}_1\Psi_m$-functions with $1 < m$ (i.e. as generalized hypergeometric functions of the so-called "Bessel" type, according to [16]).

One can mention many interesting *special cases of the multiindex (multiple) Mittag-Leffler functions.*

Example 1. A special function, generalizing classical M-L functions (1) with respect to the number of indices, was considered first by Dzrbashjan [6] in the case $\underline{m = 2}$, see also [15, App.,p.351]. He denoted it by $\Phi_{\rho_1,\rho_2}(z; \mu_1, \mu_2)$:

$$E_{(\frac{1}{\rho_1},\frac{1}{\rho_2}),(\mu_1,\mu_2)}(z) = \Phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1})\Gamma(\mu_2 + \frac{k}{\rho_2})} \tag{12}$$

and showed that it is an entire function of order $\rho = \rho_1\rho_2/(\rho_1 + \rho_2)$ and type $\sigma = (\rho_1/\rho)^{\rho/\rho_1}(\rho_2/\rho)^{\rho/\rho_2}$ with the following particular cases:

$$E_{\frac{1}{\rho},\mu}(z) = E_{(\frac{1}{\rho},0),(\mu,1)}(z) = \Phi_{\rho,\infty}(z;\mu,1); \qquad \frac{1}{1-z} = E_{(0,0),(1,1)}(z) = \Phi_{\infty,\infty}(z;1,1);$$

$$J_\nu(z) = (\frac{z}{2})^\nu E_{(1,1),(\nu+1,1)}(-\frac{z^2}{4}) = (\frac{z}{2})^\nu \Phi_{1,1}(-\frac{z^2}{4}; 1, \nu+1).$$

In [6] Dzrbashjan found also the Borel transform (in [4], [15] we called it "Borel-Dzrbashjan transform") of functions (12), some integral relations between them (representing fractional integrals with respect to separate indices) and Mellin transforms on a set of axes. The latter results allowed him to develop a theory of integral transforms in the class L_2, involving kernels related to functions (12) and further, to approximate entire functions in L_2 for an arbitrary finite system of axes starting from the origin.

We can add to the particular cases of (12) also: the so-called *Bessel-Maitland* or *Wright's functions* (misnamed by the second name Maitland of E.M. Wright):

$$\begin{aligned} J_\nu^r(z) &= {}_0\Psi_1 \left[\begin{array}{c} - \\ (\nu+1, r) \end{array} \middle| -z \right] = H^{1,0}_{0,2} \left[z \middle| \begin{array}{c} - \\ (0,1), (-\nu, r) \end{array} \right] \\ &= \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\nu + rk + 1)\, k!} = E_{(r,1),(\nu+1,1)}(z), \end{aligned} \tag{13}$$

as well as the *Struve* and *Lommel functions* (see [15, (C.8)] and for details, [17]):

$$s_{\mu,\nu}(z) = \frac{1}{4} z^{\mu+1}\, E_{(1,1),((3-\nu+\mu)/2,(3+\nu+\mu)/2)}(-\frac{z^2}{4}),$$

$$H_\nu(z) = [\pi 2^{\nu-1}(1/2)_\nu]^{-1}\, s_{\nu,\nu}(z) = \frac{1}{4} z^{\nu+1}\, E_{(1,1),(3/2,(3+2\nu)/2)}(-\frac{z^2}{4}).$$

Example 2. For arbitrary $\underline{m > 2}$: let $\underline{\forall \rho_i = \infty}$, i.e. $1/\rho_i = 0$ and $\underline{\forall \mu_i = 1}$, $i = 1, \ldots, m$. From definition (4) it is easily seen that

$$E_{(0,0,\ldots,0),(1,1,\ldots,1)}(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \, .$$

Example 3. Consider the case $m \geq 2$ with $\underline{\forall \rho_i = 1}$, $i = 1, \ldots, m$. Then

$$E_{(1,1,\ldots,1),(\mu_i+1)}(z) = {}_1\Psi_m \left[\begin{array}{c} (1,1) \\ (\mu_i, 1)_1^m \end{array} \middle| z \right] = \left[\prod_{i=1}^m \Gamma(\mu_i) \right]^{-1} {}_1F_m \left(1; \mu_1, \mu_2, \ldots, \mu_m; z\right),$$

reduces to a $\underline{{}_1F_m}$- and a to $\underline{\text{Meijer's } G^{1,1}_{1,m+1}}$-function.

Denote $\mu_i = \gamma_i + 1, i = 1, \ldots, m$ and let additionally $\underline{\text{one of } \mu_i \text{ to be } 1}$, for example: $\mu_m = 1$, i.e. $\underline{\gamma_m = 0}$. Then this multiindex M-L function becomes a *hyper-Bessel function* (in a sense of Delerue, 1953; see [15, Ch.3; App., (D.3)]):

$$J^{(m-1)}_{\gamma_i,\ldots,\gamma_{m-1}}(z) = \left(\frac{z}{m}\right)^{\sum\limits_{i=1}^{m-1} \gamma_i} E_{(1,1,\ldots,1),(\gamma_1+1,\gamma_2+1,\ldots,\gamma_{m-1}+1,1)} \left(-(\frac{z}{m})^m\right)$$

$$= \left[\prod_{i=1}^{m-1} \Gamma(\gamma_i + 1)\right]^{-1} \left(\frac{z}{m}\right)^{\sum\limits_{i=1}^{m-1} \gamma_i} {}_0F_{m-1}\left(\gamma_1 + 1, \gamma_2 + 1, \ldots, \gamma_{m-1} + 1; -(\frac{z}{m})^m\right). \tag{14}$$

In general, for $\underline{\text{rational values of } \forall \rho_i, i = 1, \ldots, m}$, functions (4) are reducible to Meijer's G-functions.

3. Generalized fractional calculus relations for multiindex M-L functions

The *Erdélyi-Kober (E-K) operators of fractional integration* (*fractional integrals*) are known generalizations of the Riemann-Liouville (R-L) fractional integrals R^δ of order $\delta > 0$, depending on two additional parameters $\gamma \in R, \beta > 0$ ([23],[15]):

$$\begin{aligned} I_\beta^{\gamma,\delta} y(z) &= [z^{-(\gamma+\delta)} R^\delta z^\gamma \; y(z^{\frac{1}{\beta}})]|_{z \to z^\beta} \\ &= \frac{z^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^z (z^\beta - t^\beta)^{\delta-1} t^{\beta\gamma} \; y(t) \; d(t^\beta) = \frac{1}{\Gamma(\delta)} \int_0^1 (1-\sigma)^{\delta-1} \sigma^\gamma \; y(z\sigma^{\frac{1}{\beta}}) \; d\sigma, \end{aligned} \tag{15}$$

with $R^\delta f(z) = z^\delta I_1^{0,\delta}$, $I_\beta^{\gamma,0} := Id$ (identity operator) and the semigroup property $I_\beta^{\gamma+\delta,\alpha} I_\beta^{\gamma,\delta} = I_\beta^{\gamma,\delta+\alpha}$, $\delta > 0, \alpha > 0$. The corresponding fractional differentiation operators, the *E-K fractional derivatives* (see [15, Ch.2] for their explicit integro-differential expression and properties), are formally written as:

$$D_\beta^{\gamma,\delta} y(z) = \left[z^{-\gamma} \; D^\delta \; z^{\gamma+\delta} \; y(z^{\frac{1}{\beta}})\right] |_{z \to z^\beta} \; . \tag{16}$$

In [15] we have developed a *generalized fractional calculus* based on generalized operators of fractional integration and differentiation that are commutative compositions of E-K fractional integrals and derivatives (15),(16), but are introduced by means of single integral operators involving G- and H-functions (8) as kernels. The effective and simple properties and succinctness of notations of the kernel special functions have allowed us to develop a comparatively full theory, encompassing many special cases, with examples of various applications in different trends and problems of analysis, differential and integral equations, etc.

DEFINITION 3. ([15],[14]) Let $m \geq 1$ be an integer; $\beta_i > 0, \gamma_i \in R, \delta_i > 0, i = 1, \dots, m$. Consider $\gamma = (\gamma_1, \dots, \gamma_m)$ as a multiweight and resp. $\delta = (\delta_1, \dots, \delta_m)$ as a *multiorder of fractional integration*. The integral operators defined as follows:

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = \begin{cases} \int\limits_0^1 H^{m,0}_{m,m} \left[\sigma \left| \begin{array}{c} (\gamma_i + \delta_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_1^m \\ (\gamma_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_1^m \end{array} \right. \right] f(z\sigma) d\sigma, \text{ if } \sum\limits_{i=1}^m \delta_i > 0, \\ f(z), \quad \text{if } \delta_1 = \delta_2 = \ldots = \delta_m = 0, \end{cases} \tag{17}$$

are said to be *multiple (m-tuple) Erdélyi-Kober fractional integration operators* and more generally, all the operators of the form

$$I f(z) = z^{\delta_0} I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) \quad \text{with} \quad \delta_0 \geq 0, \tag{18}$$

are called briefly *generalized (m-tuple) fractional integrals*.

We have considered them in different functional spaces, among them the spaces $H_\alpha(\Omega)$ introduced in next section, and have proved they map such spaces into themselves under certain conditions on the parameters. For $m = 1$, (17) turns into a classical E-K integral (15). The main feature of these single integrals involving H-functions (or G-functions in the simpler case of equal $\beta_i = \beta > 0, i = 1, \dots, m$) is that they can be equivalently represented by means of *commutative compositions of different E-K integrals* (15),

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = \left[\prod_{i=1}^m I^{\gamma_i,\delta_i}_{\beta_i} \right] f(z)$$
$$= \int\limits_0^1 \cdots \int\limits_0^1 \left[\prod_{i=1}^m \frac{(1-\sigma_i)^{\delta_i - 1} \sigma_i^{\gamma_i}}{\Gamma(\delta_i)} \right] f\left(z \sigma_1^{\frac{1}{\beta_1}} \ldots \sigma_m^{\frac{1}{\beta_m}} \right) d\sigma_1 \ldots d\sigma_m \,, \tag{19}$$

and if some of the δ_i's are zeros: $\delta_1 = \ldots = \delta_s = 0, 1 \leq s \leq m$, the corresponding multipliers $I^{\gamma_i,\delta_i}_{\beta_i} = Id$ are identity operators and the multiplicity of (17) reduces from m to $m-s$ (same for the order of the kernel H- or G-functions). The frequent appearance of compositions (19) in problems related to applications, combined with the simple but effective tools of the theory of the H- and G-functions in (17), explain the usefulness of the operators of generalized fractional calculus [15].

The *generalized (multiple E-K) fractional derivatives* $D^{(\gamma_i),(\delta_i)}_{(\beta_i),m} = \prod\limits_{i=1}^m D^{\gamma_i,\delta_i}_{\beta_i}$, corresponding to (17), are defined in [15] by means of explicit differintegral expressions:

$$D^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) := D_\eta \, I^{(\gamma_i+\delta_i),(\eta_i-\delta_i)}_{(\beta_i),m} f(z), \tag{20}$$

where the auxiliary differential operators D_η are used:

$$D_\eta = \left[\prod_{i=1}^{m} \prod_{j=1}^{\eta_i} \left(\frac{1}{\beta_j} x \frac{d}{dx} + \gamma_i + j \right) \right], \ \eta_i = \begin{cases} [\delta_i] + 1, & \text{if } \delta_i \text{ noninteger}, \\ & \text{if } \delta_i \text{ integer}, \ i = 1, \dots, m. \end{cases}$$

By analogy with the classical M-L functions (1), their multiindex analogues (4) satisfy several relations involving operators of fractional calculus (E-K operators (15),(16)) or operators of generalized fractional calculus (operators (17),(20)).

In [17] we have proved integral relations involving classical E-K fractional integrals (15).

Lemma 2. *For any complex $\lambda \neq 0$ and fixed $l : 1 \leq l \leq m, \ \alpha_l > 0$:*

$$I^{\mu_l-1,\alpha_l}_{\rho_l} E_{(\frac{1}{\rho_i}),(\mu_1,\dots,\mu_l,\dots,\mu_m)}(\lambda z)$$

$$= \frac{1}{\Gamma(\alpha_l)} \int_0^1 (1-\sigma)^{\alpha_l-1} \sigma^{\mu_l-1} \, E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z \sigma^{1/\rho_l}) d\sigma = E_{(\frac{1}{\rho_i}),(\mu_1,\dots,\mu_l+\alpha_l,\dots,\mu_m)}(\lambda z), \tag{21}$$

i.e. a fractional integration can transform a multiindex M-L function to another one with a corresponding μ_l-parameter increased by the order of integration.

Applying successively the E-K fractional integrals $I^{\mu_i-1,\alpha_i}_{\rho_i}, \alpha_i > 0, i = 1, \dots, m$ to a multiindex M-L function, on the base of (19) and (21), we obtain a relation involving a generalized fractional integral (17):

$$I^{(\mu_i-1),(\alpha_i)}_{(\rho_i),m} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) = E_{(\frac{1}{\rho_i}),(\mu_i+\alpha_i)}(\lambda z).$$

Taking in the above $\alpha_i = 1/\rho_i, i = 1, \dots, m$ and using the rules of generalized fractional calculus [15, Ch.5], we obtain the following theorem (details are in [17]).

Theorem 2. *The following relations for the multiindex Mittag-Leffler functions* (4) *in terms of the generalized fractional integrals and derivatives* (17), (20) *hold*:

$$z I^{(\mu_i-1),(1/\rho_i)}_{(\rho_i),m} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) = z E_{(\frac{1}{\rho_i}),(\mu_i+\frac{1}{\rho_i})}(\lambda z) = \frac{1}{\lambda} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) - \frac{1}{\lambda \prod_i \Gamma(\mu_i)}, \tag{22}$$

$$D^{(\mu_i-1-1/\rho_i),(1/\rho_i)}_{(\rho_i),m} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) = \lambda z \, E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) + \left[\prod_{i=1}^{m} \Gamma(\mu_i - \frac{1}{\rho_i}) \right]^{-1}. \tag{23}$$

4. Gelfond-Leontiev operators w.r.t. multiindex M-L functions

The M-L functions can be also used as generating functions of a class of the so-called Gelfond-Leontiev (G-L) operators of generalized differentiation and integration.

Definition 4. ([9]) Let the function $\varphi(\lambda) = \sum\limits_{k=0}^{\infty} \varphi_k \lambda^k$ be an entire function with a growth $(\rho > 0, \sigma \neq 0)$ such that $\lim\limits_{k\to\infty} k^{\frac{1}{\rho}} \sqrt[k]{|\varphi_k|} = (\sigma e \rho)^{\frac{1}{\rho}}$. For a function $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$, analytic in a disk $\Delta_R = \{|z| < R\}$, the correspondences

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \longmapsto D_\varphi f(z) = \sum_{k=1}^{\infty} a_k \frac{\varphi_{k-1}}{\varphi_k} z^{k-1}, \; L_\varphi f(z) = \sum_{k=0}^{\infty} a_k \frac{\varphi_{k+1}}{\varphi_k} z^{k+1} \quad (24)$$

are said to be *Gelfond-Leontiev (G-L) operators of generalized differentiation, resp. integration, with respect to the function* $\varphi(\lambda)$.

The simplest examples of G-L operators, when $\varphi(\lambda) = \exp\lambda$, i.e. $\varphi_k = 1/\Gamma(k+1), k = 0, 1, \ldots$, are the usual differentiation and integration operators: $Df(z) = \dfrac{d}{dz} f(z) = f'(z), Lf(z) = R^1 f(z) = \int\limits_0^z f(\zeta) d\zeta$.

Let $\varphi(\lambda) = E_{\frac{1}{\rho},\mu}(\lambda)$ be M-L function (1) with $\alpha = 1/\rho > 0, \beta = \mu > 0$. Then $\varphi_k = 1/\Gamma(\mu + \dfrac{k}{\rho})$, and operators (24) turn into the so-called *Dzrbashjan-Gelfond-Leontiev (D-G-L) operators of differentiation and integration*:

$$D_{\rho,\mu} f(z) = \sum_{k=1}^{\infty} a_k \frac{\Gamma(\mu + \frac{k}{\rho})}{\Gamma(\mu + \frac{k-1}{\rho})} z^{k-1}, \; L_{\rho,\mu} f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu + \frac{k}{\rho})}{\Gamma(\mu + \frac{k+1}{\rho})} z^{k+1}, \quad (25)$$

studied by Dimovski and Kiryakova [3],[4], Kiryakova [15, Ch.2].

In [15, Ch.5] we have introduced multiple (multiindex) analogues of operators (25) and studied some of their properties. It turns now that these operators are nothing but Gelfond-Leontiev operators of differentiation and integration with respect to the multiindex M-L functions (4), since the multiplier coefficients are $\varphi_k = [\prod_{i=1}^m \Gamma(\mu_i + k/\rho_i)]^{-1}$.

Definition 5. Let $f(z)$ be an analytic function in a disk $\Delta_R = \{|z| < R\}$ and $\rho_i > 0, \mu_i \in R, i = 1, ..., m$ be arbitrary parameters. The correspondences:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \;\longmapsto\; \begin{array}{l} D_{(\rho_i),(\mu_i)} f(z) = \displaystyle\sum_{k=1}^{\infty} a_k \frac{\Gamma(\mu_1 + \frac{k}{\rho_1}) ... \Gamma(\mu_m + \frac{k}{\rho_m})}{\Gamma(\mu_1 + \frac{k-1}{\rho_1}) ... \Gamma(\mu_m + \frac{k-1}{\rho_m})} z^{k-1}, \\ L_{(\rho_i),(\mu_i)} f(z) = \displaystyle\sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu_1 + \frac{k}{\rho_1}) ... \Gamma(\mu_m + \frac{k}{\rho_m})}{\Gamma(\mu_1 + \frac{k+1}{\rho_1}) ... \Gamma(\mu_m + \frac{k+1}{\rho_m})} z^{k+1} \end{array} \quad (26)$$

are called *multiple Dzrbashjan-Gelfond-Leontiev (D-G-L) differentiations and integrations*, respectively.

Let us note that $D_{(\rho_i),(\mu_i)}L_{(\rho_i),(\mu_i)}f(z) = f(z)$ for $f(z)$ analytic in Δ_R and the coincidence of the radii of convergence of $f(z)$ and of the series in (26) follows easily by the Cauchy-Hadamard formula and the asymptotic estimation of the Γ-function multipliers, like in [14],[15, Th.5.5.2]. For $m = 1$ these operators turn into D-G-L operators (25).

Denote by $\mathcal{H}(\Omega)$ the space of analytic functions in a complex domain Ω, starlike with respect to the origin $z = 0$ and consider the spaces

$$\mathcal{H}_\alpha(\Omega) = \left\{ f(z) = z^p \tilde{f}(z); p \geq \alpha, \tilde{f}(z) \in \mathcal{H}(\Omega) \right\}, \ \mathcal{H}_0(\Omega) := \mathcal{H}(\Omega), \ \alpha \in R.$$

The series representations of operators (26) can be analytically continued for analytic functions in starlike domains $\Omega \supset \Delta_R$ by means of single or repeated integral or differintegral expressions, as special cases of the operators (17),(20) of the generalized fractional calculus.

THEOREM 3. *Let $\Delta_R \subset \Omega, \mu_i \geq 0, i = 1, \ldots, m;\ \alpha := \max\limits_{1 \leq k \leq m} \{-\mu_k \rho_k\} \leq 0$. Then, the multiple D-G-L integration operator* (26) *can be analytically continued from $\mathcal{H}(\Delta_R)$ into $\mathcal{H}_\alpha(\Omega)$ by means of the single integral operator*

$$L_{(\rho_i),(\mu_i)}f(z) = z \int\limits_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{array}{c} (\mu_i, \frac{1}{\rho_i})_1^m \\ (\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i})_1^m \end{array} \right. \right] f(z\sigma)d\sigma = z\, I^{(\mu_i - 1),(\frac{1}{\rho_i})}_{(\rho_i),m} f(z), \tag{27}$$

that is, by a generalized fractional integral of form (17). The multiple D-G-L derivative (26) has also a differintegral representation in terms of (20), especially for analytic functions in $\mathcal{H}(\Omega) \supset \mathcal{H}(\Delta_R)$,

$$D_{(\rho_i),(\mu_i)}f(z) = z^{-1}\, D^{(\mu_i - 1 - \frac{1}{\rho_i}),(\frac{1}{\rho_i})}_{(\rho_i),m} f(z) - \left[\prod_{i=1}^m \frac{\Gamma(\mu_i)}{\Gamma(\mu_i - \frac{1}{\rho_i})}\right] \frac{f(0)}{z}. \tag{28}$$

In the case $m = 1$, (27) and (28) are the analytical continuations of the D-G-L operators (25) as E-K fractional integrals and derivatives, found in [3],[4]:

$$L_{\rho,\mu}f(z) = z^1\, I_\rho^{\mu-1,\frac{1}{\rho}} f(z) = \frac{z}{\Gamma(\frac{1}{\rho})} \int\limits_0^1 (1-\sigma)^{\frac{1}{\rho}-1} \sigma^{\mu-1} f(z\sigma^{\frac{1}{\rho}})d\sigma,$$

$$D_{\rho,\mu}f(z) = z^{-1} D_\rho^{\mu-1-\frac{1}{\rho},\frac{1}{\rho}} f(z) - \frac{f(0)\Gamma(\mu)}{\Gamma(\mu - \frac{1}{\rho})} z^{-1}.$$

In the same papers transmutation operators relating R-L fractional integrals $R^{\frac{1}{\rho}}$ and D-G-L generalized integrations $L_{\rho,1}, L_{\rho,\mu}$ have been obtained, and by their help, also a family of convolutions of $L_{\rho,\mu}$. By means of these convolutions we have given explicit representations of the commutant of $L_{\rho,\mu}$, that is, of all the linear operators commuting with it in $\mathcal{H}(\Omega)$.

EXAMPLE 4. Let us note now the special case when $m \geq 2$ arbitrary, but all $\rho_i = 1, i = 1, \ldots, m$, and denote $\mu_i := \gamma_i + 1, i = 1, \ldots, m$. Then multiple D-G-L operators (26), resp. (27) and (28), are nothing but *hyper-Bessel integral and differential operators* L, B (with their parameter "β" taken as $\beta = 1$):

$$Lf(z) := L_{(1),(\gamma_i+1)} f(z) = z \int_0^1 G_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\gamma_i + 1)_1^m \\ (\gamma_i)_1^m \end{matrix} \right. \right] f(z\sigma) d\sigma = z \, I_{(1),m}^{(\gamma_i),(1)} f(z), \tag{29}$$

$$D_{(1),(\gamma_i+1)} f(z) = z^{-1} \, D_{(1),m}^{(\gamma_i-1),(1)} \, f(z) - \frac{[\prod_{i=1}^m \gamma_i] \, f(0)}{z} = Bf(z) - \frac{c}{z} \, ,$$

$$Bf(z) := z^{-1} \, D_{(1),m}^{(\gamma_i-1),(1)} \, f(z) = D_{(1),m}^{(\gamma_i),(1)} \, z^{-1} \, f(z), \tag{30}$$

introduced by Dimovski, 1966 and studied by Dimovski and Kiryakova in a list of works like [1], [15, Ch.3]. More precisely, the multiple D-G-L differential operators $D_{(1),(\gamma_i+1)}$ differ from the hyper-Bessel differential operators (30) by terms of the form c/z, and the constant c is zero, if at least one of the γ_i's is zero. Since in this case the multiindex M-L function (4) generating the multiple D-G-L operators (29),(30) is exactly the hyper-Bessel function (14) of Example 3, we can conclude the following: *the hyper-Bessel integral and differential operators* (29), (30), *can be viewed also as multiple D-G-L operators of integration and differentiation with respect to hyper-Bessel functions* (14).

Now, comparing generalized fractional integrals and derivatives (27),(28) with these involved in relations (22),(23) one easily obtains *integral and differential equations satisfied by* (4), *written in terms of multiple D-G-L operators.*

LEMMA 3. *The multiple Mittag-Leffler functions* (4) *satisfy the following relations* ($\lambda \neq 0$) :

$$L_{(\rho_i),(\mu_i)} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) = \frac{1}{\lambda} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) - \frac{1}{\lambda \prod_i \Gamma(\mu_i)} \, , \tag{31}$$

$$D_{(\rho_i),(\mu_i)} E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z) = \lambda \, E_{(\frac{1}{\rho_i}),(\mu_i)}(\lambda z). \tag{32}$$

Formula (32) can be seen also as a *differential equation of fractional order (i.e. "multiorder"*$(1/\rho_1, \ldots, 1/\rho_m)$), *satisfied by multiindex M-L functions* (4).

Extending the results of Dimovski [1] for the hyper-Bessel integral operators (29), and using a theorem of Luchko and Jakubovich [19] (Theorem 2.14 in [18]) for convolutions of a class of generalized fractional integrals (17) with $\forall \ \delta_i \beta_i = \mu, \ i = 1, \ldots, m$ (here $\mu = 1$), we can write down the following *family of convolutions* of the multiple D-G-L integrals (26),(27) in the sense of Dimovski [2].

THEOREM 4. *For $\lambda \geq \max\limits_i(\mu_i\rho_i - 1) \geq -1$, the operations $(\overset{\lambda}{\star})$, defined by means of generalized fractional integrals of form* (17) :

$$(f \overset{\lambda}{\star} g)(z) = z^{\lambda+1} I_{(\rho_i),m}^{(2\mu_i-1),(\frac{\lambda+1}{\rho_i}-\mu_i)} (f \circ g)(z) \tag{33}$$

and of auxiliary operation

$$(f \circ g)(x) = \int\limits_0^1 \dots \int\limits_0^1 \prod_{i=1}^m [t_i(1-t_i)]^{\mu_i-1} f\left[z \prod_{i=1}^m t_i^{\frac{1}{\rho_i}}\right] \; g\left[z \prod_{i=1}^m (1-t_i)^{\frac{1}{\rho_i}}\right] dt_1 \dots dt_m, \tag{34}$$

are convolutions of the multiple D-G-L integrations (26), (27) *in $\mathcal{H}_\alpha(\Omega)$ as well as in the subspace $\mathcal{H}(\Omega)$.*

This result allows to find convolutional representations of the operator $L = L_{(\rho_i),(\mu_i)}$ itself, as well as of all the linear operators M that commute with it in $\mathcal{H}(\Omega)$: $ML = LM$ (the so-called *commutant* of L); as well as to develop other elements of the operational calculus for the multiple D-G-L operators.

Examples of convolutional products of some "basic functions", when for simplicity we take $\lambda = -1$, are:

$$z^p \overset{-1}{\star} z^q = z^{p+q} \prod_{i=1}^m \frac{\Gamma(\mu_i + p/\rho_i)\Gamma(\mu_i + q/\rho_i)}{\Gamma(\mu_i + (p+q)/\rho_i)}, \quad p, q > \alpha,$$

$$E_{(\frac{1}{\rho}_i),(\mu_i)}(\alpha z) \overset{-1}{\star} E_{(\frac{1}{\rho}_i),(\mu_i)}(\beta z) = \frac{\alpha E_{(\frac{1}{\rho}_i),(\mu_i)}(\alpha z) - \beta E_{(\frac{1}{\rho}_i),(\mu_i)}(\beta z)}{(\alpha - \beta) \prod\limits_{i=1}^m \Gamma(\mu_i)}, \quad \alpha, \beta \neq 0.$$

5. Laplace type integral transform

In [4],[15, Ch.2] we have shown that the role of a Laplace transformation for the D-G-L operators $D_{\rho,\mu}, L_{\rho,\mu}$ $(m = 1)$ can be played by the *Borel-Dzrbashjan transform*:

$$\mathcal{B}_{\rho,\mu}\left\{\sum_{k=0}^\infty a_k z^k\right\} = \sum_{k=0}^\infty \frac{a_k \Gamma(\mu + \frac{k}{\rho})}{s^{k+1}};$$

$$\mathcal{B}_{\rho,\mu}\{f(z); s\} = \rho s^{\mu\rho-1} \int_0^\infty \exp(-s^\rho z^\rho) z^{\mu\rho-1} f(z) dz, \tag{35}$$

usually considered in ρ-convex domains like $\Omega = \mathcal{D}_\rho(0;\nu) = \left\{s \colon \Re(s^\rho) > \nu, |\arg s| < \frac{\pi}{2\rho}\right\}$. There, for $\frac{1}{2} < \mu < \frac{1}{\rho}$ and $\mathcal{L} := \mathcal{L}_\rho(0;\nu) = \partial D_\rho(0;\nu)$ the following *complex inversion formula* holds (see [7]):

$$f(z) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} E_{\frac{1}{\rho},\mu}(sz) \; \mathcal{B}_{\rho,\mu}\{f; s\} \, ds, \quad \nu > \nu_0. \tag{36}$$

We have proved that (35) has the same convolution as the D-G-L integration operators, taken with $\lambda = -1$ (see operation (33), $m = 1$), namely:

$$\mathcal{B}_{(\rho),(\mu)} \left\{ f \overset{-1}{\star} g; s \right\} = \mathcal{B}_{(\rho),(\mu)} \left\{ f(z); s \right\} \ . \ \mathcal{B}_{(\rho),(\mu)} \left\{ g(z); s \right\}, \tag{37}$$

and have found some basic operational properties as well as images and convolutions of some particular functions.

Here we introduce a *Laplace type integral transform, corresponding to the multiple D-G-L operators.*

Definition 6. The Laplace type H-transform, defined by

$$\mathcal{B}(s) = \mathcal{B}_{(\rho_i),(\mu_i)} \left\{ f(z); s \right\} \tag{38}$$

$$= \int\limits_0^\infty H^{m,0}_{0,m} \left[sz \middle| \begin{array}{c} - \\ (\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i}) \end{array} \right] f(z) dz = \frac{1}{s} \int\limits_0^\infty H^{m,0}_{0,m} \left[sz \middle| \begin{array}{c} - \\ (\mu_i, \frac{1}{\rho_i}) \end{array} \right] \frac{f(z)}{z} dz$$

is said to be a *multiple Borel-Dzrbashjan (B-D) transform*, corresponding to multiple D-G-L operators (26)-(28).

We consider (38) for functions

$$\begin{gathered} f(z) \in \mathcal{H}_\alpha,\ \alpha = \max_i(-\mu_i \rho_i);\ f(z) = \mathcal{O}\left\{ \exp(\lambda r^{1/\lambda} z^{1/\lambda}) \right\} \quad \text{as} \quad |z| \to \infty; \\ \text{and for} \quad s \in \left\{ \Re(s^{1/\lambda}) > \nu_0; |\arg s| < \tfrac{\pi\lambda}{2} \right\}, \end{gathered} \tag{39}$$

with some $\nu_0 \in R$ and $r := \left[\prod\limits_{i=1}^m \rho_i^{1/\rho_i} \right]$, $\lambda := \sum\limits_{i=1}^m \frac{1}{\rho_i} > 0$.

The restriction for the exponential growth of the originals $f(z)$ as $|z| \to \infty$, follows from the known asymptotic behaviour of the kernel $H^{m,0}_{0,m}$-function, namely: it vanishes exponentially as $|z| \to \infty$, see [24, (2.2.14), p.13],[15, (E.17$'$)].

Example 5. If we put $m = 1$ in (38), we obtain the "single" B-D transform (35). Moreover, if $\mu = \rho = 1$, then the *Laplace integral transform* follows as a special case, which justifies the name *Laplace type integral transform* given to (38).

From definition (38) one can easily evaluate the images of some basic original functions $f(z)$. The aparatus of the H-functions is used.

Example 6. Since

$$\mathcal{B}_{(\rho_i),(\mu_i)}\{z^q\} = s^{-(q+1)} \ [\prod_{i=1}^m \Gamma(\mu_i + \frac{q}{i})],\ q > \alpha = \max_{1 \le i \le m} (-\mu_i \rho_i), \tag{40}$$

then for $\mu_i > 0, \rho_i > 0, i = 1, \ldots, m$ we obtain the *image of an analytic function*

$$f(z) = \sum_{k=0}^m a_k z^k \in \mathcal{H}(\Delta_R) \ \overset{\mathcal{B}_{(\rho_i),(\mu_i)}}{\longmapsto} \ \mathcal{B}(s) = \sum_{k=0}^m \frac{a_k \prod_{i=1}^m \Gamma(\mu_i + k/\rho_i)}{s^{k+1}} \ . \tag{41}$$

Example 7. It is interesting to note that the *image of multiindex M-L function* (4) under (38) is an elementary fraction, typical for the operational calculus:

$$\mathcal{B}_{(\rho_i),(\mu_i)}\left\{E_{(\frac{1}{\rho}_i),(\mu_i)}(z);s\right\}=\frac{1}{s-1}\,. \tag{42}$$

The multiple B-D transform, as defined by (38),(41), plays the same role for the multiple D-G-L derivatives and integrals like the Laplace transform $\mathcal{L}\{f(z);s\}$ for the classical operators of differentiation and integration. First, we have for function (4) the usefull relations (32),(42), analogous to the well-known ones for the exponential function:

$$\frac{d}{dz}\exp(\lambda z)=\lambda\exp(\lambda z),\quad \mathcal{L}\{\exp z;s\}=\frac{1}{s-1}\,.$$

Next, it is easy to prove that (38) algebrizes the multiple D-G-L derivatives and integrals, i.e. reduces them to multiplications by fixed rational functions.

Theorem 5. *If $f(z)$ and $s\in C$ satisfy conditions* (39), *then the multiple D-G-L integration operator* (26), (27) *is algebrized by the multiple B-D transform* (38) :

$$\mathcal{B}_{(\rho_i),(\mu_i)}\left\{L_{(\rho_i),(\mu_i)}f(z);s\right\}=\frac{1}{s}\,\mathcal{B}_{(\rho_i),(\mu_i)}\{f(z);s\} \tag{43}$$

and

$$\mathcal{B}_{(\rho_i),(\mu_i)}\left\{D_{(\rho_i),(\mu_i)}f(z);s\right\}=s\,\mathcal{B}_{(\rho_i),(\mu_i)}\{f(z);s\}\;-\;f(0)\;[\prod_{i=1}^{m}\Gamma(\mu_i)]. \tag{44}$$

The latter relation generalizes the well-known differential law for the Laplace transform. The *proof* follows in an elementary way by replacing (38),(27) into L.H.S. of (43) and the widely used techniques of *evaluation of integrals of products of two H-functions* (see [15, (E.21$'$)],[24, (5.1.1), p.58]). The additional term from (28), depending on $f(0)$, appears now in a modified form in (44).

For the purposes of an operational calculus, to deal with the multiple B-D transform, one should dispose with some inversion formulas. To compare with similar Laplace type transforms, like B-D transform (35) or the more general, so-called *Obrechkoff transform* ([20],[1],[15, Ch.3]), we can look for various complex inversion formulas, either of the kind of (36), or by means of Mellin transform techniques, shown to be more effective. The following complex inversion formula seems the most useful one.

Theorem 6. *If $f(z)$, satisfying* (39), *has a B-D image* (38) :

$$\mathcal{B}(s):=\mathcal{B}_{(\rho_i),(\mu_i)}\{f(z);s\}\,,$$

then the inversion formula

$$f(z)=\int\limits_{c-i\infty}^{c+i\infty}\frac{z^{-q}}{\prod_{i=1}^{m}\Gamma(\mu_i-q/\rho_i)}\;dq\;\left\{\int\limits_{0}^{\infty}s^{-q}\;\mathcal{B}(s)ds\right\} \tag{45}$$

holds, provided that the integrals

$$\int_0^\infty z^{c-1}\, f(z)dz\ ,\ \int_0^\infty s^{-q}\, \mathcal{B}(s)ds$$

are absolutely convergent for $q = c + iT$, $-\infty < T < \infty$ and c is a suitably chosen constant, $c < -\alpha$.

The *proof* is quite similar to that of Th.3.9.12, [15] in the case of Obrechkoff transform. Its effectiveness can be easily tested on Examples 6,7.

Another complex inversion formula of contour integral type (36), with a kernel-function related to $E_{(\frac{1}{\rho_i}),(\mu_i)}(sz)$ can be looked for, too.

The operation (33), being a convolution of D-G-L integration operator $L_{(\rho_i),(\mu_i)}$, could be proven to be also a convolution of the multiple B-D transform (38), with a relation similar to (37).

The details of all the proofs in this section, as well as another representaion of the kernel-function of (38), relationship between (38) and the m-dimensional Laplace transform, etc. we leave for another paper.

Example 8. The Laplace type integral transform (38), corresponding to multiple D-G-L operators (29),(30), i.e. to the hyper-Bessel integral and differential operators, is obtained for $\rho_1 = ... = \rho_m = 1$ and reduces to a G-transform. This is nothing but the so-called *Obrechkoff integral transform*

$$\mathcal{O}\{f(z);s\} = \beta s^{-\beta(\gamma_m+1)+1} \int_0^\infty G^{m,0}_{0,m}\left[(sz)^\beta \,\middle|\, \begin{matrix} - \\ (\gamma_i + 1 - \frac{1}{\beta})_1^m \end{matrix}\right] f(z)dz, \qquad (46)$$

when $\beta = 1$ is taken.

Transformation (46) was first introduced by Obrechkoff [20] in 1958 and studied by Dimovski [1] in the form

$$\mathcal{O}\{f(z);s\} = \beta \int_0^\infty z^{\beta(\gamma_m+1)-1}\, K\left[(sz)^\beta\right] f(z)dz, \qquad (47)$$

with a kernel given as

$$K(z) = \int_0^\infty \dots \int_0^\infty \exp\left(-u_1 - \dots - u_{m-1} - \frac{z}{u_1 \dots u_{m-1}}\right) \prod_{i=1}^m u_i^{\gamma_i - \gamma_i - 1} du_1 \dots du_{m-1}.$$

It was shown to be a suitable Laplace type integral transform for building an operational calculus for hyper-Bessel operators (29),(30). Convolutions, differential properties and inversion formulas were found. Later, its new representation of form (46), by means of a $G^{m,0}_{0,m}$-function as the kernel $K(z)$, allowed a more detailed and easy study, see [15, Ch.3].

6. Conclusions

• Examples 3,4,8 (with all $\rho_i = 1, i = 1, \ldots, m$) reveal the *importance of the multiple D-G-L operators not only as examples of generalized (multiple E-K) fractional integrals and derivatives involving H-functions, but also as further generalizations of the widely used hyper-Bessel operators.* The hyper-Bessel differential operators (30) are usually encountered in equations of mathematical physics and other applicable topics, in the forms

$$B = z^{\alpha_0}\left(\frac{d}{dz}\right) z^{\alpha_1}\left(\frac{d}{dz}\right) z^{\alpha_2} \ldots \left(\frac{d}{dz}\right) z^{\alpha_m} = z^{-\beta} P_m\left(z\frac{d}{dz}\right), \; 0 < z < \infty, \quad (48)$$

with m-th degree polynomials P_m. In [15, Ch.3] we have represented them and their linear right inverse operators, the hyper-Bessel integral operators L, as generalized fractional derivatives and integrals of "integer" multiorders $(\delta_1, \ldots, \delta_m) = (1, 1, \ldots, 1)$, as in (29),(30). The hyper-Bessel functions of Delerue (14) (in Example 3, following as a special case of multiindex M-L functions (4)), are closely related to hyper-Bessel operators (29),(30),(48) and the solutions of the hyper-Bessel differential equations $By(x) = \lambda y(x)$ are represented in their terms. Thus, the multiple D-G-L operators of differentiation $D_{(\rho_i),(\mu_i)}$ are natural extensions, a kind of "fractional" analogues (of multiorder of differentiation $(1/\rho_1, \ldots, 1\rho_m)$ instead of (1,...,1)) of the hyper-Bessel differentiations (48).

• The multiindex Mittag-Leffler functions (4) can be seen as *"fractional indices" analogues of the hyper-Bessel functions*, and the multiple Borel-Dzrbashjan integral transforms (38) (being H-transforms) - as *"fractional indices" analogues of the Obrechkoff transforms* (46) (being G-transforms).

• In Introduction we have complained about ignoring, till recently, of the classical Mittag-Leffler functions as known inverse Laplace transforms, in the handbooks on special functions and tables of Laplace transforms. For example, in the existing tables, only M-L functions (1) with indices $\alpha, \beta = \frac{1}{2}, 1, 2$, etc., i.e. the exponential and trigonometric functions, the error functions, incomplete gamma functions, etc. could be found. This has been one of the reasons, for the applied scientists, to avoid an appropriate treating of problems of the real world, by means of fractional order modelling differential equations, as such a necessity arises e.g. in the control theory when systems with fractional order transfer functions are considered ([21]). To deal successfully with fractional order dynamic systems, one needs to know explicitly the inverse Laplace transforms $\mathcal{L}^{-1}$ of s-functions like

$$\frac{s^{\beta-1}}{s^\beta + \lambda}, \; \frac{s^{\alpha-\beta}}{s^\beta + \lambda}, \; \ldots, \; G_n(s) = \frac{1}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}} + \ldots + a_0 s^{\beta_0}}, \quad (49)$$

$\beta_n > \beta_{n-1} > \ldots > \beta_0 > 0$. Recently, many authors have interpeted explicitly such elements of operational calculus as functions of Mittag-Leffler type, as Podlubny in [21] by means of the functions $(k = 0, 1, 2, \ldots)$:

$$y(z) = \mathcal{E}_k(z, \pm\lambda; \alpha, \beta) := z^{\alpha k + \beta - 1} E^{(k)}_{\alpha,\beta}(\lambda z^\alpha), \quad \text{with} \quad \mathcal{L}\{y(z); s\} = \frac{k! s^{\alpha-\beta}}{(s^\alpha \mp \lambda)^{k+1}}.$$

What could be the use of the multiindex Mittag-Leffler functions considered in this survey, *for treating problems modelled by more general fractional "multi-order" differential or integral equations*? One possible hint is given by the "multiple Borel-Dzrbashjan image" (42), as an *element of the operational calculus* based upon the H-transform (38).

Another, *possibly easier visible interpretation* (in Mikusinski's / Dimovski's frameworks), could be found in the *survey by Luchko* [18], same volume. Considering *operational calculi for a class of multiple E-K fractional integrals and derivatives* very close to multiple D-G-L operators (27),(28), he proposes representations of some elements of the convolution quotient field by elements of the initial ring of functions. There, in Section 2, formulas (49),(50) appear that involve the multiindex M-L functions, called *"M-L functions of vector index"*, namely:

$$\frac{I}{S-\kappa} = z^{\mu-\lambda}\, E_{(1-\alpha_i+a_i(\mu-\lambda)),(a_i\mu)}(\kappa z^{\mu}) \tag{50}$$

with some $\kappa \in R$, $\mu > 0$, λ as in Th.4, $\sum_{i=1}^{m} \alpha_i > 0$, similarly for the elements $1/(S-\kappa)^n$.

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